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On the chromatic uniqueness of certain bipartite graphs

Y.H. Peng

Department of Mathematics, Universiti Pertanian Malaysia, 43400 Serdang, Selangor D.E., Malaysia

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Abstract

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Let $K(p, q)$, $p \leqslant q$, denote the complete bipartite graph in which the two partite sets consist of p and q vertices, respectively. We denote by $K^{-r}(p, q)$, the family of all graphs obtained by deleting any r distinct edges from $K(p, q)$. Teo and Koh showed that $K^{-1}(p, q)$ is chromatically unique (in short χ-unique) for all p, q such that $3 \leqslant p \leqslant q$. In this paper, we obtain a sufficient condition for a graph in $K^{-2}(p, q)$ to be χ-unique. Using this result, we then prove that each graph in $K^{-2}(p, p+d)$ is χ-unique for $p \geqslant 4$ and $0 \leqslant d \leqslant 3$. For $d \geqslant 4$, the graphs in $K^{-2}(p, q+d)$ are χ-unique if $p > (A+\sqrt{B})/4d^2$, where A and B are polynomials in d. We also show that each graph ($\not\cong K(4,4) - K(1,3)$) in $K^{-3}(p, p+d)$ is χ-unique, for $p \geqslant 4$ and $d = 0, 1$; and all graphs in $K^{-3}(p, p+2)$ are χ-unique if and only if all graphs in $K^{-4}(p+1, p+1)$ are χ-unique, where $p \geqslant 4$. Finally we prove that every graph in $K^{-4}(p, p+1)$ is χ-unique for $p \geqslant 5$.

1. Introduction

All graphs considered in this paper are finite, undirected, simple and loopless. For a graph G, we denote by $P(G; \lambda)$, or simply by $P(G)$, the chromatic polynomial of G (see [11–12]). Two graphs G and H are said to be *chromatically equivalent*, or simply *χ-equivalent*, in notation, $G \sim H$ if $P(G) = P(H)$. A graph G is said to be *chromatically unique*, or in short *χ-unique*, if $P(H) = P(G)$ implies that H is isomorphic ($\cong$) to G. Numerous families of χ-unique graphs were discovered (see [2, 4–5, 7–9, 14]) since this notion was first introduced in 1978 by Chao and Whitehead [3]. Here, we shall be concerned with bipartite graphs, i.e., graphs whose vertex set can be partitioned into two subsets U and W such that every edge of the graph joins a vertex in U to a vertex in W.

Let $K(p, q)$, $p \leqslant q$, denote the complete bipartite graph where p and q are the numbers of vertices in its two partite sets. Salzberg et al. [13] proved that the

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graph $K(p, q)$ is χ-unique if $p \geqslant 2$ and $0 \leqslant q-p \leqslant \max\{5, \sqrt{2p}\}$; and conjectured that the graph $K(p, q)$ is χ-unique for all p, q with $2 \leqslant p \leqslant q$. In [15], Tomescu showed that the graph $K(p, q)$ is a χ-unique if $p \geqslant 2$ and $0 \leqslant q-p \leqslant 2\sqrt{p+1}$. And very recently, Teo and Koh [14] proved that the conjecture is true.

Salzberg et al. also proved in [13] that the graph $K^{-1}(p, q)$ obtained by deleting any edge from $K(p, q)$ is χ-unique if $p \geqslant 3$ and $0 \leqslant q-p \leqslant 1$. Teo and Koh showed in [14] that the graph $K^{-1}(p, q)$ is χ-unique for all p, q with $3 \leqslant p \leqslant q$. In the same paper, the authors proposed the following problem: Study the chromatic uniqueness of the family of graphs $K^{-2}(p, q)$, obtained by deleting any two distinct edges from $K(p, q)$. It is the purpose of this paper to present some results related to the problem. We also examine the chromatic uniqueness of the bipartite graphs obtained by deleting any three or four distinct edges from $K(p, q)$. Our approach here is similar to that of Salzberg et al. [13].

For terms used but not defined here we refer to [1].

2. Preliminary results and notation

Let $V(G)$, $E(G)$, and $\chi(G)$ denote, respectively, the vertex set, edge set and chromatic number of a graph G. Let $N_Q(G)$ and $N_K(G)$ denote, respectively, the number of pure quadrilaterals (cycles of order four without chords) in G and the number of complete subgraphs with four vertices of G. We begin with the following necessary conditions for two graphs to be χ-equivalent.

Theorem 1. *Let G and H be two χ-equivalent graphs. Then*:

(i) $|V(G)| = |V(H)|$;
(ii) $|E(G)| = |E(H)|$;
(iii) $\chi(G) = \chi(H)$;
(iv) $N_Q(G) - 2N_K(G) = N_Q(H) - 2N_K(H)$;
(v) *G is connected if and only if H is connected*;
(vi) *G is 2-connected if and only if H is 2-connected.*

Remark. Result (iv) in the theorem above was obtained by Farrell [6] while the result (vi) can be found in Woodall [18] and Whitehead and Zhao [16].

Farrell [6] has given explicit expressions in terms of subgraphs of the graph for the first five coefficients of the chromatic polynomial of a graph. As a consequence of his result [6, Theorem 2] and Theorem 1 above, we have the following.

Theorem 2. *If H is χ-equivalent to G, and G is a bipartite graph, then H is also a bipartite graph having the same number of vertices, edges, pure quadrilaterals, and complete bipartite subgraphs $K(2, 3)$ as G.*

Throughout this paper, the following notation will be used: We shall denote by $G(p, q)$ any bipartite graph with $|U| = p$ and $|W| = q$. We shall always assume that $p \leqslant q$, unless stated otherwise. Thus, given $G = G(p, q)$ with $U = \{u_1, u_2, \ldots, u_p\}$ and $W = \{w_1, w_2, \ldots, w_q\}$, the *simplified adjacency matrix of* G is the $p \times q$ matrix $[\varepsilon_{ij}]$, where

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } u_i w_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

The degrees of u_i and w_i are denoted by d_i and e_i, respectively. Given a bipartite graph $G = G(p, q)$, we shall denote by $\bar{G}$, the complement of G in $K(p, q)$, and write $\bar{\varepsilon}_{ij} = 1 - \varepsilon_{ij}$. The degrees of u_i and w_i in $\bar{G}$ will be denoted by $\bar{d}_i$ and $\bar{e}_i$, respectively.

We shall denote by $K^{-r}(p, q)$ the family of graphs obtained by deleting any r distinct edges from the complete bipartite graph $K(p, q)$. (The U-ends of the deleted edges will be coloured black when they are expressed diagrammatically as in Fig. 1.) Thus, the family $K^{-2}(p, q)$ consists of the following three bipartite graphs:

$$K_1^{-2}(p, q) = K(p, q) - 2K_2,$$
$$K_2^{-2}(p, q) = K(p, q) - K_u(1, 2),$$
$$K_3^{-2}(p, q) = K(p, q) - K_w(1, 2),$$

where $K_u(1, 2)$ and $K_w(1, 2)$ denote any two edges of $K(p, q)$ which are incident at a vertex of U and W, respectively. We show the graphs $G_i = K_i^{-2}(3, 4)$ $(i = 1, 2, 3)$ in Fig. 1. We sometimes use the same notation to denote $K^{-r}(p, q)$ and its members, if there is no likelihood of confusion.

A complete bipartite graph with any r edges deleted, $K^{-r}(p, q)$ $(p \leqslant q)$, has $p + q$ vertices and $pq - r$ edges. Therefore, according to Theorem 2, a possible candidate that can be χ-equivalent to $K^{-r}(p, q)$ is a bipartite graph $G = G(m, n)$ $(m \leqslant n)$ satisfying $m + n = p + q$ and $mn \geqslant pq - r$. By solving these two expressions for m and n, we obtain the following theorem.

Theorem 3. *If G is χ-equivalent to $K^{-r}(p, q)$, then $G = G(p + k, q - k)$ for some k, $-h \leqslant k \leqslant \frac{1}{2}(q - p)$ where h is the largest nonnegative integer satisfying*

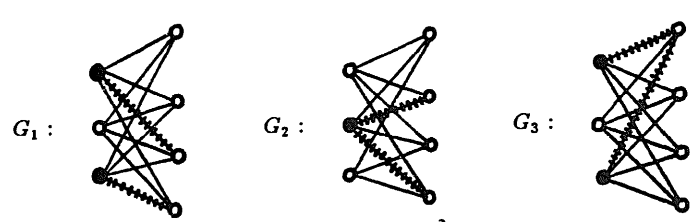

Fig. 1. The graphs in $K^{-2}(3, 4)$.

$(q-p)h+h^2 \leqslant r$. In this case, G is obtained from the complete bipartite graph $K(p+k, q-k)$ by deleting $t_k=(q-p)k-k^2+r$ edges.

3. Complete bipartite graphs with two edges deleted

The following lemma will be useful in proving the chromatic uniqueness of the complete bipartite graph with two edges deleted.

Lemma 1. *If $K_i^{-2}(p,q) \not\cong K_j^{-2}(p,q)$ for $p \geqslant 3$ and $1 \leqslant i < j \leqslant 3$, then $K_i^{-2}(p,q) \not\sim K_j^{-2}(p,q)$.*

Proof. It can easily be confirmed that

$$N_Q(K_1^{-2}(p,q)) = \binom{p}{2}\binom{q}{2} - 2(p-1)(q-1)+1,$$

$$N_Q(K_2^{-2}(p,q)) = \binom{p}{2}\binom{q}{2} - 2(p-1)(q-1)+(p-1), \quad \text{and}$$

$$N_Q(K_3^{-2}(p,q)) = \binom{p}{2}\binom{q}{2} - 2(p-1)(q-1)+(q-1).$$

Thus, $N_Q(K_i^{-2}(p,q)) \neq N_Q(K_j^{-2}(p,q))$ if $K_i^{-2}(p,q) \not\cong K_j^{-2}(p,q)$ for $p \geqslant 3$ and $1 \leqslant i < j \leqslant 3$. By Theorem 2, the result follows. □

Theorem 4. *The graph $K_i^{-2}(p,q)$ $(i=1,2,3)$ is χ-unique for $p \geqslant 4$ and $q-p=0$ or 1.*

Proof. By Theorem 3, the only graphs that can possibly be χ-equivalent to $K_i^{-2}(p,q)$ $(i=1,2,3)$ are $K_j^{-2}(p,q)$ $(j=1,2,3)$ and $K^{-1}(p-1,q+1)$ (if $q-p=0$) or $K(p-1,q+1)$ (if $q-p=1$). Since $K^{-1}(p-1,q+1)$ and $K(p-1,q+1)$ are χ-unique (see [14]) and not isomorphic to $K_i^{-2}(p,q)$, Lemma 1 implies the result. □

According to Theorems 2 and 3, to prove that $K_i^{-2}(p,q)$ $(i=1,2,3)$ with $q \geqslant p+2$ is χ-unique, it suffices to show that $N_Q(G(p+k,q-k)) \neq N_Q(K_i^{-2}(p,q))$ $(i=1,2,3)$ for every bipartite graph $G(p+k,q-k)$ with $0 \leqslant k \leqslant \frac{1}{2}(q-p)$. To this end, we need the following formula for $N_Q(G)$, where G is any bipartite graph $G(p+k,q-k)$:

$$\begin{aligned} N_Q(G) = {} & \tfrac{1}{4}(p+k)(p+k-1)(q-k)(q-k-1) \\ & - \tfrac{1}{4}[(2p+2k-1)(2q-2k-1)\bar{s} - (2p+2k-1)\bar{D} \\ & - (2q-2k-1)\bar{E} + 4\bar{F} - \bar{J} - 2\bar{s}^2], \end{aligned} \tag{1}$$

where $\bar{s} = |E(\bar{G})|$, $\bar{D} = \sum \bar{d}_i^2$, $\bar{E} = \sum \bar{e}_j^2$, $\bar{F} = \sum_{i,j} \bar{d}_i \bar{e}_j \bar{\varepsilon}_{ij}$ and

$$\bar{J} = \sum_{i,i',j,j'} \bar{\varepsilon}_{ij} \bar{\varepsilon}_{ij'} \bar{\varepsilon}_{i'j} \bar{\varepsilon}_{i'j'}.$$

The formula above was obtained by Salzberg et al. [13], who used the principle of inclusion–exclusion in their calculation. For any positive integers m and n, we write

$$T(m, n) = \{4mn - 6m - 6n + 6, 4mn - 10m - 6n + 14, 4mn - 6m - 10n + 14\}.$$

Theorem 5. *The graph $K_i^{-2}(p, q)$ $(i = 1, 2, 3)$, $q \geqslant p + 2$, is χ-unique if for every bipartite graph $G(p + k, q - k)$ having $pq - 2$ edges, with $0 \leqslant k \leqslant \frac{1}{2}(q - p)$, the following inequalities hold*:

$$\begin{aligned} \Sigma = {} & t_k(2pq - p - q - 4 + t_k) - (2p + 2k - 1)\bar{D} \\ & - (2q - 2k - 1)\bar{E} + 4\bar{F} - \bar{J} \neq \xi, \end{aligned} \tag{2}$$

where $\xi \in T(p, q)$ and $t_k = (q - p)k - k^2 + 2$.

Proof. It is a straightforward consequence of (1), Theorems 2 and 3, and the fact that $t_k = (q - p)k - k^2 + 2$. $\square$

Note. By Lemma 1, we do not have to examine (2) for $k = 0$.

The following simple lemma (see [14, Lemma 2]) will improve the applicability of Theorem 5.

Lemma 2. *Deleting all isolated vertices from $\bar{G}$ does not change the values of $\bar{s}$, $\bar{D}$, $\bar{E}$, $\bar{F}$ and $\bar{J}$.*

Theorem 6. *The graph $K_i^{-2}(p, p + 2)$ $(\not\cong K_3^{-2}(3, 5))$ is χ-unique for $p \geqslant 3$ and $1 \leqslant i \leqslant 3$.*

Proof. By Theorem 5 and Lemma 1, the graph $K_i^{-2}(p, p + 2)$ $(i = 1, 2, 3)$ is χ-unique if the inequalities (2) hold for every bipartite graph $G(p + 1, p + 1)$ obtained by deleting three edges from $K(p + 1, p + 1)$. By Lemma 2, we find the values of Σ for each graph $\bar{H}_i$ $(1 \leqslant i \leqslant 6)$ exhibited in Fig. 2. It is easy to confirm that the expressions for Σ are as shown in Table 1.

Now it is straightforward to show that $\Sigma \neq \xi$, for each $\xi \in T(p, p + 2)$ and for any integer $p \geqslant 4$, except that when $p = 6$, we have $6(p^2 - 3p + 1) = 4p^2 - 4p - 6$. This exceptional case means that $N_Q(K_1^{-2}(6, 8)) = N_Q(K(7, 7) - K(1, 3))$; so the graph $K_1^{-2}(6, 8)$ is not χ-unique if it is χ-equivalent to $K(7, 7) - K(1, 3)$. By Theorem 3 and Lemma 1, the only possible non-isomorphic χ-equivalent candidate for $K_1^{-2}(6, 8)$ is $K(7, 7) - K(1, 3)$. But we can easily verify that the

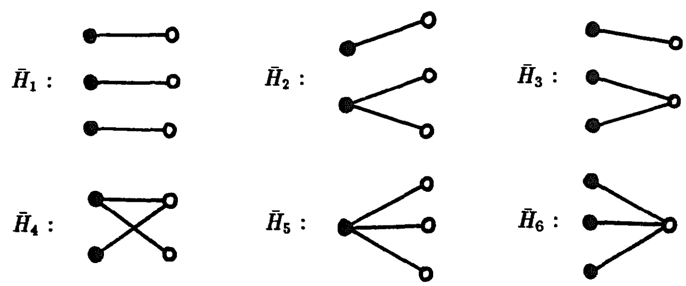

Fig. 2. Three deleted edges from $K(p+1, p+1)$.

numbers of complete bipartite subgraphs $K(2,3)$ in $K_1^{-2}(6,8)$ and $K(7,7)-K(1,3)$ are 1060 and 1119, respectively. So by Theorem 2, $K_1^{-2}(6,8) \not\sim K(7,7)-K(1,3)$, and hence $K_1^{-2}(6,8)$ is χ-unique.

Similarly, when $p=3$ and $i \neq 3$, we have $\Sigma \neq \xi$ for each $\xi \in T(p, p+2)$, except that $2(3p^2-7p+3) = 2(2p^2-2p-3)$. This exceptional case means that $N_Q(K_1^{-2}(3,5)) = N_Q(K(4,4)-P_4)$. Theorem 5 implies that $K_2^{-2}(3,5)$ is χ-unique. It remains to show that $K_1^{-2}(3,5)$ is also χ-unique. By Theorem 3 and Lemma 1, the only possible non-isomorphic χ-equivalent candidate for $K_1^{-2}(3,5)$ is $K(4,4)-P_4$. By using the *reduction formula* due to Whitney [17], we obtain

$$P(K_1^{-2}(3,5)) - P(K(4,4)-P_4) = \lambda^3 - 3\lambda^2 + 2\lambda.$$

This implies that $K_1^{-2}(3,5) \not\sim K(4,4)-P_4$. Thus $K_1^{-2}(3,5)$ is χ-unique.

The proof is now complete. □

Note. The bipartite graphs $K_1^{-2}(3,5)$ and $K(4,4)-P_4$ satisfy all the necessary conditions in Theorem 2 but they are not χ-equivalent. The graph $K_3^{-2}(3,5)$ is not χ-unique because it is χ-equivalent to the non-isomorphic graph $K(4,4)-K(1,3)$.

Theorem 7. *The graph $K_i^{-2}(p, p+3)$ $(\not\equiv K_3^{-2}(3,6))$ is χ-unique for $p \geqslant 3$ and $1 \leqslant i \leqslant 3$.*

Proof. By Theorem 5 and Lemma 1, the graph $K_i^{-2}(p, p+3)$ $(i=1,2,3)$ is χ-unique if the inequalities (2) hold for every bipartite graph $G(p+1, p+2)$, obtained by deleting four edges from $K(p+1, p+2)$. By Lemma 2, we find the values of Σ for each graph $\bar{H}_i$ $(1 \leqslant i \leqslant 16)$ exhibited in Fig. 3. The expressions of Σ for each $\bar{H}_i$ $(1 \leqslant i \leqslant 16)$ as shown in Table 2 can easily be computed.

Table 1
$\Sigma = 3(2p^2+2p-3) - (2p+1)(\bar{D}+\bar{E}) + 4\bar{F} - \bar{J}$.

$\bar{H}_i$	Σ
$\bar{H}_1$	$6(p^2-p-1)$
$\bar{H}_2$, $\bar{H}_3$	$2(3p^2-5p-1)$
$\bar{H}_4$	$2(3p^2-7p+3)$
$\bar{H}_5$, $\bar{H}_6$	$6(p^2-3p+1)$

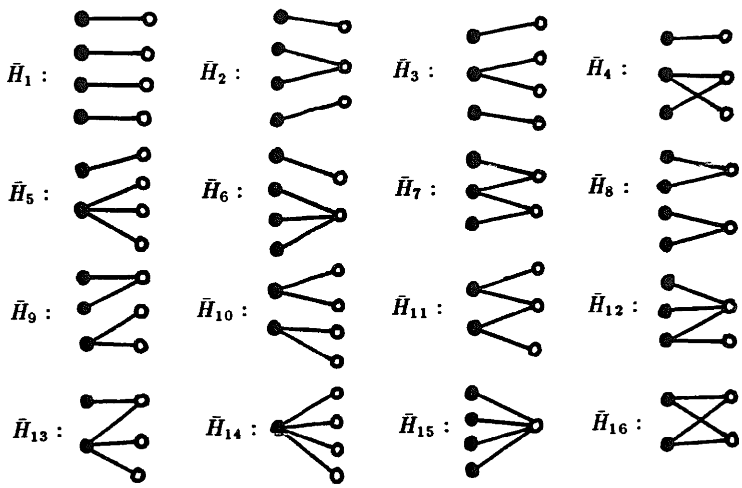

Fig. 3. Four deleted edges from $K(p+1, p+2)$.

It is now straightforward to show that $\Sigma \neq \xi$ for each $\xi \in T(p, p+3)$ and for each integer $p \geqslant 3$ except for the following five cases.

When $p = 3$,

(i) $8p^2 - 12p - 16 = 4p^2 - 4p - 4$ which means $N_Q(K_{\bar 2}^{-2}(3, 6)) = N_Q(K(4, 5) - \bar H_6)$;

(ii) $8p^2 - 16p - 4 = 4p^2 - 4p - 4$ which means $N_Q(K_{\bar 2}^{-2}(3, 6)) = N_Q(K(4, 5) - \bar H_{12})$;

(iii) $8p^2 - 24p + 8 = 4p^2 - 4p - 16$ which means $N_Q(K_{\bar 3}^{-2}(3, 6)) = N_Q(K(4, 5) - \bar H_{14})$; and when $p = 5$,

(iv) $8p^2 - 24p + 8 = 4p^2 - 12$ which means $N_Q(K_{\bar 1}^{-2}(5, 8)) = N_Q(K(6, 7) - \bar H_{14})$;

(v) $8p^2 - 24p - 16 = 4p^2 - 4p - 16$ which means $N_Q(K_{\bar 3}^{-2}(5, 8)) = N_Q(K(6, 7) - \bar H_{15})$.

The theorem is now established if the graph $K_{\bar 2}^{-2}(3, 6)$, $K_{\bar 1}^{-2}(5, 8)$ and $K_{\bar 3}^{-2}(5, 8)$ can be shown to be χ-unique. To do this, we proceed as follows: By Theorem 3 and Lemma 1, the only possible χ-equivalent candidates for $K_{\bar 2}^{-2}(3, 6)$, $K_{\bar 1}^{-2}(5, 8)$ and $K_{\bar 3}^{-2}(5, 8)$ are $K(4, 5) - \{\bar H_6$ or $\bar H_{12}\}$, $K(6, 7) - \bar H_{14}$ and $K(6, 7) - \bar H_{15}$, respectively. Theorem 1(vi) implies that $K_{\bar 2}^{-2}(3, 6) \nsim K(4, 5) - \{\bar H_6$ or $\bar H_{12}\}$. By Theorem 2, $K_{\bar 1}^{-2}(5, 8) \nsim K(6, 7) - \bar H_{14}$ and $K_{\bar 3}^{-2}(5, 8) \nsim K(6, 7) - \bar H_{15}$ since the

Table 2
$\Sigma = 3(2p^2 + 2p - 3) - (2p + 1)(\bar D + \bar E) + 4\bar F - \bar J$

$\bar H_i$	Σ		$\bar H_i$	Σ
$\bar H_1$	$8p^2 - 16$		$\bar H_9$	$8p^2 - 8p - 12$
$\bar H_2$	$8p^2 - 4p - 16$		$\bar H_{11}$	$8p^2 - 12p$
$\bar H_3$	$8p^2 - 4p - 12$		$\bar H_{12}$	$8p^2 - 16p - 4$
$\bar H_4$, $\bar H_{10}$	$8p^2 - 8p - 8$		$\bar H_{13}$, $\bar H_{16}$	$8p^2 - 16p - 4$
$\bar H_5$, $\bar H_7$	$8p^2 - 12p - 4$		$\bar H_{14}$	$8p^2 - 24p + 8$
$\bar H_6$	$8p^2 - 12p - 16$		$\bar H_{15}$	$8p^2 - 24p - 16$
$\bar H_8$	$8p^2 - 8p - 16$			

numbers of complete bipartite subgraphs $K(2,3)$ in $K_1^{-2}(5,8)$, $K_3^{-2}(5,8)$, $K(6,7)-\bar{H}_{14}$ and $K(6,7)-\bar{H}_{15}$ are 597, 630, 595 and 615, respectively. □

Note. The graph $K_3^{-2}(3,6)$ is not χ-unique because it is χ-equivalent to the non-isomorphic graph $K(4,5)-\bar{H}_{14}$.

We shall now investigate the chromatic uniqueness of the graphs in $K^{-2}(p,q)$ for $q-p\geqslant 4$, by analyzing some deeper consequences of (2).

Lemma 3. *The following bounds for the terms of* (2) *hold*:
(1) $0\leqslant t_k\leqslant \frac{1}{4}(q-p)^2+2$;
(2) $t_k\leqslant \bar{D}\leqslant t_k^2$;
(3) $t_k\leqslant \bar{E}\leqslant t_k^2$;
(4) $t_k\leqslant \bar{F}\leqslant t_k^3$;
(5) $t_k\leqslant \bar{J}\leqslant t_k^2$.

Proof. The proof is similar to that of Lemma 1 in [13]. □

According to Lemma 3, after replacing the positive terms in Σ of (2) by their lower bounds, and the regative ones by their upper bounds, we obtain

$$\Sigma\geqslant t_k(2p^2+2pd-2p-d)-t_k(4p+2d-2),$$

where $d=q-p$ and $t_k=dk-k^2+2$. Note that $\max T(p,p+d)=4p^2+4pd-12p-6d+6$. So by Theorem 5, the graph in $K^{-2}(p,p+d)$ with $d\geqslant 4$ is χ-unique if

$$t_k(2p^2+2pd-2p-d)-t_k^2(4p+2d-2)>4p^2+4pd-12p-6d+6 \quad (3)$$

holds for $1\leqslant k\leqslant d/2$.

It can easily be checked that (3) is satisfied if

$$p>\left\{\frac{2t_k^2(d-1)+t_kd-6(d-1)}{2t_k-4}+\left[\frac{2t_k^2-t_k(d-1)+2(d-3)}{2t_k-4}\right]^2\right\}^{1/2} + \frac{2t_k^2-t_k(d-1)+2(d-3)}{2t_k-4}. \quad (4)$$

Since the right hand side of the inequality (4) increases as k increases from $k=1$ to $k=d/2$, we can conclude that (3) holds if

$$p>\left\{\frac{2\left(\frac{d^2}{4}+2\right)^2(d-1)+\left(\frac{d^2}{4}+2\right)d-6(d-1)}{2\left(\frac{d^2}{4}+2\right)-4} + \left[\frac{2\left(\frac{d^2}{4}+2\right)^2-\left(\frac{d^2}{4}+2\right)(d-1)+2(d-3)}{2\left(\frac{d^2}{4}+2\right)-4}\right]^2\right\}^{1/2}$$

$$+\frac{2\left(\frac{d^2}{4}+2\right)^2-\left(\frac{d^2}{4}+2\right)(d-1)+2(d-3)}{2\left(\frac{d^2}{4}+2\right)-4}. \tag{5}$$

Therefore, after simplifying (5) we have proved the following.

Theorem 8. *The graph $K_i^{-2}(p,p+d)$ $(d\geqslant 4)$ is χ-unique for $1\leqslant i\leqslant 3$ and $p>(A+\sqrt{B})/4d^2$, where $A=d^4-2d^3+18d^2+32$ and $B=d^8+36d^6+324d^4+1088d^2+1024$.*

Remark. If d is odd, then the lower bound for p in the theorem above can slightly be improved by substituting $(d^2+7)/4$ for t_k in (4).

Example. The graph $K_i^{-2}(p,p+d)$ $(i=1,2,3)$ is χ-unique if $p\geqslant 16$ (resp., 20, 25, 31, 38, 46, 55, 65, 76, 88, 101, 115) and $d=4$ (resp., 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15).

We end this section with the following conjecture.

Conjecture. The graph $K_i^{-2}(p,q)$ $(\not\cong K_3^{-2}(3,q)$ with $q\geqslant 5)$ is χ-unique for $1\leqslant i\leqslant 3$ and all p, q such that $3\leqslant p\leqslant q$.

4. Complete bipartite graphs with three edges deleted

The family of graphs, $K^{-3}(p,q)$, obtained by deleting any 3 distinct edges from $K(p,q)$ consists of the following six bipartite graphs $K_i^{-3}(p,q)$ $(1\leqslant i\leqslant 6)$ defined by

$$K_i^{-3}(p,q)=K(p,q)-\bar{H}_i,$$

where $\bar{H}_i$ are the graphs exhibited in Fig. 2.

Lemma 4. *If $K_i^{-3}(p,q)\not\cong K_j^{-3}(p,q)$ for $p\geqslant 4$ and $1\leqslant i<j\leqslant 6$, then $K_i^{-3}(p,q)\not\sim K_j^{-3}(p,q)$.*

Proof. By using (1), it is easy to see that if $K_i^{-3}(p,q)\not\cong K_j^{-3}(p,q)$ for $p\geqslant 4$ and $1\leqslant i<j\leqslant 6$, then $N_Q(K_i^{-3}(p,q))\neq N_Q(K_j^{-3}(p,q))$, except for $q=3p-4$, $i=3$, $j=5$; and $q=2p-1$, $i=4$, $j=5$. For these two exceptional cases, if $p\geqslant 4$, we have

$$\begin{aligned}&N_{K(2,3)}(K_5^{-3}(p,3p-4))-N_{K(2,3)}(K_3^{-3}(p,3p-4))\\&\quad=3p^2-13p+14\neq 0\end{aligned}$$

and

$$N_{K(2,3)}(K_5^{-3}(p,2p-1)) - N_{K(2,3)}(K_4^{-3}(p,2p-1))$$
$$= p^2 - 3p + 2 \neq 0.$$

Thus by Theorem 2, the result follows. □

Theorem 9. *The graph $K_i^{-3}(p,q)$ ($\not\cong K_5^{-3}(4,4)$) is χ-unique for $1 \leqslant i \leqslant 6$, $p \geqslant 4$ and $q-p=0$ or 1.*

Proof. We consider two cases.

Case 1. $q-p=0$.

By Theorem 3, the only possible χ-equivalent candidates for $K_i^{-3}(p,q)$ are $K_j^{-2}(p-1,q+1)$ $(1 \leqslant j \leqslant 3)$ and $K_l^{-3}(p,q)$ $(1 \leqslant l \leqslant 6)$. But by Lemma 4, we need only consider $K_j^{-2}(p-1,q+1)$ $(1 \leqslant j \leqslant 3)$. The graph $K_j^{-2}(p-1,q+1)$ $(1 \leqslant j \leqslant 3)$ is χ-unique if $K_j^{-2}(p-1,q+1) \not\cong K_3^{-2}(3,5)$ (see Theorem 6). Therefore $K_i^{-3}(p,q)$ ($\not\cong K_5^{-3}(4,4)$) is χ-unique for $p \geqslant 4$ and $1 \leqslant i \leqslant 6$. Note that $K_5^{-3}(4,4)$ is χ-equivalent to the non-isomorphic $K_3^{-2}(3,5)$.

Case 2: $q-p=1$.

By Theorem 3, the only possible χ-equivalent candidates for $K_i^{-3}(p,q)$ are $K^{-1}(p-1,q+1)$ and $K_j^{-3}(p,q)$ $(1 \leqslant j \leqslant 6)$. From Lemma 4, we do not need to consider $K_j^{-3}(p,q)$ $(1 \leqslant j \leqslant 6)$. Also by the result of Teo and Koh [14], the graph $K^{-1}(p-1,q+1)$ is χ-unique for $p \geqslant 4$. Thus, the graph $K_i^{-3}(p,q)$ is χ-unique for $p \geqslant 4$ and $1 \leqslant i \leqslant 6$. □

5. Complete bipartite graphs with four edges deleted

The graph obtained by deleting any four distinct edges from $K(p,q)$ is one of the following sixteen bipartite graphs $K_j^{-4}(p,q)$ $(1 \leqslant j \leqslant 16)$ defined by

$$K_j^{-4}(p,q) = K(p,q) - \bar{H}_j,$$

where $\bar{H}_j$ are the graphs shown in Fig. 3.

The following lemma can easily be confirmed by counting the numbers of pure quadrilaterals and complete bipartite subgraphs $K(2,3)$, and by applying Theorem 2.

Lemma 5. *Let G and H be any graphs in $K^{-4}(p,p)$ with $p \geqslant 5$. If $G \not\cong H$, then $G \not\sim H$.*

Note. The case when $G = K_9^{-4}(p,p)$ and $H = K_8^{-4}(p,p)$ follows from Example 1 in [10].

The next result follows immediately from Theorem 3, Lemmas 4 and 5, and the fact that $K(p, q)$ is χ-unique for all p, q such that $2 \leqslant p \leqslant q$ (see [14]).

Theorem 10. *Let $p \geqslant 4$ be any integer. Then all six graphs $K_i^{-3}(p, p+2)$ $(1 \leqslant i \leqslant 6)$ are χ-unique if and only if all sixteen graphs $K_j^{-4}(p+1, p+1)$ $(1 \leqslant j \leqslant 16)$ are χ-unique.*

Lemma 6. *Let G and H be any graphs in $K^{-4}(p, p+1)$ with $p \geqslant 5$. If $G \not\cong H$, then $G \not\sim H$.*

Proof. By using (1), it is not difficult to verify that if $K_i^{-4}(p, p+1) \not\cong K_j^{-4}(p, p+1)$ for $p \geqslant 7$ and $1 \leqslant i < j \leqslant 16$, then $N_Q(K_i^{-4}(p, p+1)) \neq N_Q(K_j^{-4}(p, p+1))$, except for the following three pairs $(i, j) = (4, 10)$, $(5, 7)$ and $(13, 16)$. By the principle of inclusion–exclusion, it is easy to show that

$$N_{K(2,3)}(K_4^{-4}(p, p+1)) \neq N_{K(2,3)}(K_{10}^{-4}(p, p+1))$$

and

$$N_{K(2,3)}(K_5^{-4}(p, p+1)) \neq N_{K(2,3)}(K_7^{-4}(p, p+1)).$$

Similarly, for $p = 5$ and 6, and $1 \leqslant i < j \leqslant 16$, $N_Q(K_i^{-4}(p, p+1)) \neq N_Q(K_j^{-4}(p, p+1))$ if $K_i^{-4}(p, p+1) \not\cong K_j^{-4}(p, p+1)$, except for the following three pairs $(i, j) = (8, 11)$ when $p = 5$, and $(i, j) = (6, 13)$ and $(6, 16)$ when $p = 6$. Using the same principle, the numbers of complete bipartite subgraphs $K(2, 3)$ in $K_8^{-4}(5, 6)$, $K_{11}^{-4}(5, 6)$, $K_6^{-4}(6, 7)$, $K_{13}^{-4}(6, 7)$, and $K_{16}^{-4}(6, 7)$ can easily be computed and they are 144, 145, 540, 544, and 544, respectively. It was shown in [10, Example 2] that $K_{13}^{-4}(p, p+1)$ is not chromatically equivalent to $K_{16}^{-4}(p, p+1)$, for $p \geqslant 4$.

By Theorem 2, the lemma now follows. □

Theorem 11. *The graph $K_i^{-4}(p, p+1)$ is χ-unique for $p \geqslant 5$ and $1 \leqslant i \leqslant 16$.*

Proof. This follows from Lemma 6 and Theorems 3 and 7. □

Note. The graphs $K_i^{-4}(4, 5)$ for $i = 6, 7, 8, 12$ and 14 are not χ-unique since $K_6^{-4}(4, 5)$, $K_7^{-4}(4, 5)$ and $K_{14}^{-4}(4, 5)$ are χ-equivalent to the non-isomorphic graphs $K_{12}^{-4}(4, 5)$, $K_8^{-4}(4, 5)$ and $K_3^{-2}(3, 6)$, respectively. For $i = 1, 2, 3, 4, 5, 9, 10, 11, 13, 15$, and 16, the graphs $K_i^{-4}(4, 5)$ are χ-unique.

We end this paper with the following problem.

Problem. Let G and H be two bipartite graphs. If $G \sim H$, prove (or disprove) that $N_{K(m,n)}(G) = N_{K(m,n)}(H)$ for all m, n such that $2 \leqslant m \leqslant n$.

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